*polyhedral surfaces, discrete curvatures,
shape analysis, 3D modelling, segmentation.*

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3D MODELLING AND SEGMENTATION WITH DISCRETE CURVATURES

Recent concepts of discrete curvatures are very important for Medical and Computer Aided Geometric Design applications. A first reason is the opportunity to handle a discretisation of a continuous object, with a free choice of the discretisation. A second and most important reason is the possibility to define second-order estimators for discrete objects in order to estimate local shapes and manipulate discrete objects. There is an increasing need to handle polyhedral objects and clouds of points for which only a discrete approach makes sense. These sets of points, once structured (in general meshed with simplexes for surfaces or volumes), can be analysed using these second-order estimators. After a general presentation of the problem, a first approach based on angular defect, is studied. Then, a local approximation approach (mostly by quadrics) is presented. Different possible applications of these techniques are suggested, including the analysis of 2D or 3D images, decimation, segmentation... We finally emphasise different artefacts encountered in the discrete case.

1. INTRODUCTION

This paper offers a review of work on discrete curvatures for Computer Aided Geometric Design applications and a discussion about some ideas for further studies. We consider the discrete curvatures computed on a set of points and their relevance to continuous curvatures when the density of the points increases. First of all, it is interesting to recall that curvatures are fundamental tools for studying curves and surfaces. These geometrical invariants are basic elements of the theoretical study from a differential point of view (see for example [17]). The combination of the two principal curvatures makes it possible to obtain relevant information on classes of surfaces. The most encountered combinations are the Gaussian and mean curvatures, but other combinations exist. Their calculation is easy if the evaluation of derivatives is possible. Their use for shape estimation or optimisation, in spite of their effectiveness, quickly implies a high number of operations. Classically, one can try to switch from continuous to discrete values, much easier to compute and handle. This corresponds to the first interest of discrete curvatures. It is significant to note that within this framework, the selected discretisation is free. It can thus be adapted, subject to the definition of control criteria so as to obtain coherent results with continuous notions. But discrete curvatures have another more general and important application. There is an increasing need to handle polygonal or polyhedral objects and even clouds of points where only a discrete approach makes sense. These sets of points, once structured (ordered for the curves, and in general meshed with simplexes for surfaces or volumes), can be analysed using these second-order estimators. The problem has rather old origins, since one finds its first elements in the works of Gauss and Legendre ([16]). The first recent work on the subject was proposed by Alexandrov ([6]).

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A general presentation of the problem is given in next section. The approach based on angular defect is studied in section 3. Then, the local approximation approach is presented in section 4. Section 5 finally emphasises different artefacts encountered in the discrete case, suggesting forthcoming studies to obtain reliable tools in any case.

2. THE PROBLEM AND ITS CONSTRAINTS

A discrete geometrical object is a polyhedral surface defined by a set of points (vertices) and a structure (neighbourhood, relationship between vertices). Information interpretable as a curvature of the surface must be found. Classically, the envelope of the cloud of points is triangulated to provide the structure giving the neighbourhood information. Otherwise, only the k nearest neighbours of vertex P are computed, leading to poorer neighbourhood information. In both cases, wrong triangulations or neighbourhoods can be obtained for irregular discretisations of complex objects. By wrong triangulations, we mean either triangulations with self-intersections, triangulations with holes or inappropriate with respect to the initial object, inhomogeneous triangulations, ... Even if important works have been devoted to the improvement of triangulations, all the difficulties cannot be avoided without additional information on the object geometry.

It is sometimes assumed that a normal vector N is known or can be easily computed, yielding the possibility of considering the neighbours of S as points of a surface $z=h(x,y)$ expressed in the Darboux frame, or even in the global frame. Assuming that a normal vector is known before the local analysis or receiving a rather good one as a result from this analysis is an important issue. We are in favour of the second one, which is more general and avoids difficulties generally associated with irregular discretisations. In fact, curvature information, primarily of second order, is closely related to the determination of the normal vector which is of first order, and also related to the considerations of length, angle and area. If the Gaussian curvature K is intrinsic, mean curvature H can be determined only with the normal vector N, which emphasises the requirement for a global solution for the 3 objects (N, K, H), or for the approximated tangent plane which is closely related.

One can assert a priori several requirements for discrete curvature computations, both on the method and on the results. Mainly, by decreasing order of importance

- behaviour: the result must at least be invariant by isometries and be modified in a “reasonable” way by affine transformations,
- convergence: the result must converge to the continuous curvatures when the number of points tends to infinity; roughly speaking, the distance between points tends to 0,
- local behaviour: it does not seem reasonable to use “long distance” information to find properties which are primarily analogous to derivatives,
- independence of the position and structure of vertices: For a surface it should be a foreseeable modification during a change of triangulation, even if the latter influences the result.

There are mainly two approaches for discrete curvature computations:

- define the property by taking the limit in the continuous case and take the definition before reaching the limit in discrete cases (see next section).
- find a simpler geometrical object which satisfies certain conditions and use known properties for this object (see section 4).

3. THE ANGULAR DEFECT APPROACH

3.1. BACKGROUND ON CURVATURES IN THE CONTINUOUS CASE

Let X be a smooth surface in $\Re^3$ (and let $(u,v) \rightarrow X(u,v)$ be a local parameterisation of this surface). Let S be the *Weingarten map* of the tangent plane of X. k_1 and k_2, eigenvalues of S are called the *principal curvatures* of X at P and its eigenvectors are called the *principal directions*. The *mean curvature*, and Gaussian curvature are respectively defined by: $H(P) = (k_1 + k_2)/2$, $K(P) = k_1.k_2$ and characterise the local shape of X around P (convex, concave or saddle). However, Gaussian curvature can be defined in another way. Let $N : X \rightarrow \Re^3$ be the smooth unit normal vector field on S, $N : X \rightarrow S^2$ (where S^2 denotes the unit sphere of $\Re^3$) is called the Gauss map. Given U an open neighbourhood of P, let us define the quantity:

$$K(P) = \lim_{U \to P} \frac{Area(N(U))}{Area(U)} \qquad (1)$$

Gauss *Theorem* states that $K(P)$ so defined actually equals Gaussian curvature at P.

In order to be compatible with continuous notions, it is straightforward that the discrete Gaussian curvature does not vanish only for the vertices of the polyhedral surface and that the discrete mean curvature does not vanish only along the edges of this surface (figure 1).

Now in the case of discrete surfaces a suitable equivalent for these notions can also be defined. The following formula is called the “classical formula for Gaussian curvature” at vertex P:

$$K = \frac{2\pi - \sum \alpha_i}{\frac{1}{3}\sum area(T_i)} \qquad (2)$$

where the numerator is the area of Gauss indicatrix and the $1/3$ weighting is intuitively explained by sharing the area between the three vertices of a triangle.

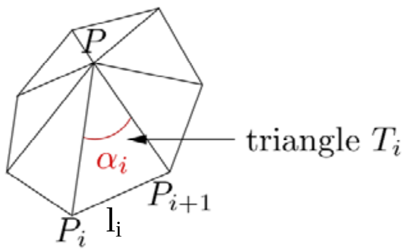

Fig. 1. Vertex P and its neighbourhood

3.2. MAIN RESULTS

The results obtained by Borrelli and Boix ([9,10,11]) provide some of the most significant theoretical developments. They relate more precisely the notions of neighbourhood, angles, length, areas, Gaussian and mean curvature and provide a theoretical framework for the intuitive formula presented above. These works start from a study of the case of geodesic triangles, and the approximation of the chord length of a curve plotted on a smooth surface according to the corresponding arc length and reciprocally. For a triangle drawn on a smooth surface M, a formula applying spherical trigonometry and expressing the difference between α (the angle of geodesics) and α’ (the angle of chords) is available. Assuming that the vertex is an umbilical point, this formula can be simplified and leads to an approximation of K called the *corrected* formula, which considers the length l_i of triangle edges opposed to the angles α_i and the area of T_i (see figure 1):

$$K = \frac{2\pi - \sum \alpha_i}{\frac{1}{2}\sum area(T_i) - \frac{1}{8}\sum \cot\mathrm{an}(\alpha_i) l_i^2} \tag{3}$$

The mean curvature is classically attached to an edge e. In [9] and [2], the authors present the following classical formula for the integral mean curvature of e:

$$H(e)=1/2\ \alpha.length(e) \tag{4}$$

where $\alpha(e)$ is the diedral angle between the neighbouring faces and $l(e)$ denotes the length of the edge e. Bousquet ([12]) gives a definition for a mean curvature located at the vertices. Intuitively, considering that this curvature is evenly split among both vertices of e, one obtains the following approximate for the mean curvature at vertex P:

$$H=1/4\ \alpha.length(e) \tag{5}$$

In [26], Meyer, Desbrun and al. present a new derivation of these discrete curvature invariants using averaging Voronoï cells. However, as the Voronoï cell does not make sense in the presence of obtuse angles, they introduce a measure called “mixed area” based on Voronoï when possible and on barycenters otherwise. The area of such a mixed cell is denoted by A_{Mixed}. Then, the mean curvature normal operator is defined by:

$$H(P).N(P) = \frac{1}{2A_{Mixed}} \sum_i (\cot\mathrm{an}(\alpha_i) + \cot\mathrm{an}(\beta_i))\ (P - P_i) \tag{6}$$

where $N(P)$ denotes the unit normal vector at P. The other notations are illustrated in figure 2. Therefore, this formula also provides a value for $N(P)$. The Gaussian curvature operator is given by:

$$K(P) = \frac{2\pi - \sum \alpha_i}{A_{Mixed}} \tag{7}$$

Other formulae are also provided for principal curvatures and principal directions. However, convergence of these operators has still to be explored: their quality is only emphasised through a number of numerical tests followed by applications to the denoising and enhancement of meshes.

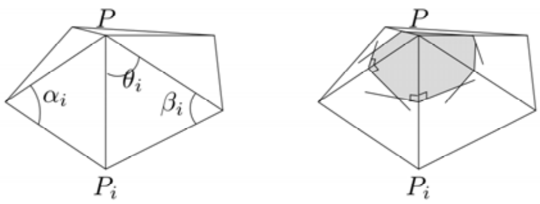

Fig. 2. Voronoï area around vertex P, A_{mixed} in grey

The main theoretical results about the convergence of discrete curvatures are presented in [19,14,11].

In [20] and [15], the authors establish the *convergence in measure* of discrete curvatures towards continuous ones (i.e. by integrating on an open set), under the assumption that the triangles have a limited thickness, that is, do not tend to be flattened. This condition implies that the number of vertex neighbours is bounded. However, the question of the *simple convergence* of the formulae on angular defects is more delicate. Actually, simple convergence of the classical formula cannot be ensured. In [11], simple convergence of the corrected formula is studied in details. The convergence is mainly reached for valence 6 regular triangulations and for valence 4 regular vertex with the one-ring neighbours aligned with the principal directions. Another problem in this frame is that the denominator of the corrected formula (called the module of the mesh at P) which can be interpreted as a fraction of area can be negative for triangles far from equilateral. This problem illustrates one of the difficulties encountered for very irregular triangulations (flat triangles, vertices with great valence).

3.3. APPLICATIONS

The main applications of discrete curvatures certainly concern *fairing and enhancement of meshes* and *segmentation of data*. The ideas exposed in the following can be applied to any type of data, particularly in the field of medical images which has been less investigated.

The first important works to cite are those of Alboul and Van Damme ([2,3]) dealing with the quality of a given triangulation and methods to improve it. They first define an energy measure of the triangulation. A significant contribution of these works is to propose a *vertex classification* into three classes. As a matter of fact, according to the arrangement of the vertices, the Gaussian indicatrix is a possibly self-intersecting oriented loop. It is either a loop with trigonometrically or counter clockwise (or positive) orientation, or a loop with clockwise (or negative) orientation, or several loops with partly positive and negative orientations. It is thus possible to calculate a positive part D^+ and a negative part D^- of the angular defect, which correspond to the areas of the positive and negative loops on the unit sphere. The *absolute discrete curvature*, defined as $D^+ + D^-$, is the discrete analogue of $|k_1|+|k_2|$ in the continuous case and $\sum D^+ + D^-$ (called *total absolute curvature*) is used as energy criterion. The authors suggest a classification of polyhedron vertices into three classes called *convex*, *saddle* and “*mixed*” (figure 3). We prefer labelling the latter “fan” by analogy with the obtained shape.

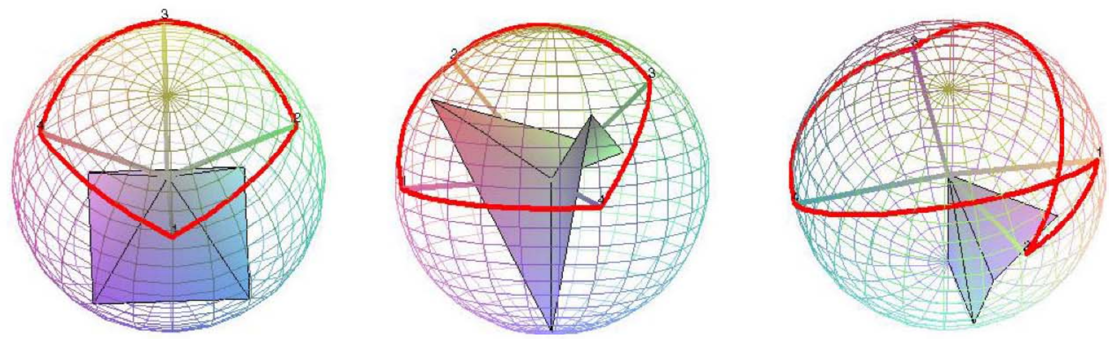

Fig. 3. Convex, saddle and fan vertices and their Gaussian indicatrix

In [1,5] the authors recall that unlike 2D triangulations, any two 3D triangulations are not equivalent under the flip operation: self-intersections may occur during this process. Therefore, they extend their work to meshes containing self-intersections. As a result, new types of intermediate vertices appear (called *pinch vertices* in [1]). In [4], starting from Alboul and Van Damme' works, the authors compare various energy functions for optimising a triangulation. In particular, they consider energy measures such as *jump in normal derivatives* or total absolute curvature, but also a function defined as a combination of both Gaussian and mean curvatures ($4H^2$-$2K$) over all the vertices. In [19], the authors develop and implement the idea suggested in [2,3], using three energy criteria: norms 1 and 2 of mean curvature and norm 1 of absolute curvature.

A discrete variational approach is proposed in [12,13] for improving the quality of B-spline surfaces. A variational method whose objective function is strain energy of the surface is introduced. Control points can be free of fixed. In order to avoid local minima, a simulated annealing algorithm is used. This approach obviously involves large computing times. To improve performances, a discrete objective function is introduced. It is computed on the control polyhedron (previously triangulated). It is defined starting from strain energy E_{ij} in each pole by:

$$E = \sum_{i,j} E_{ij} \quad \text{with } E_{ij} = 4H_{ij}^2 - 2K_{ij} \tag{8}$$

The Gaussian curvature is computed with the *classical formula*. The mean curvature on an edge is computed with (4). The value H_{ij} is obtained by summing the contributions of all the incident edges at vertex P_{ij} applying (5). The decrease of computing time is obviously very significant (in a ratio up to 50 to 60 percent on the studied examples). It is especially interesting to observe that the results with discrete and continuous optimisations are comparable in practice, although no precautions for convergence have been taken.

A series of papers ([21,22,23,28]), explore fairing of meshes and define the *Kobbelt umbrella operator*. Actually, given $X : \rightarrow \Re^3$ a parameterisation of a smooth surface, the so called "thin plate" energy of X is defined by:

$$\varepsilon_{TP}(X) = \int \left(\frac{\partial^2 X}{\partial x^2}\right)^2 + 2\left(\frac{\partial^2 X}{\partial x \partial y}\right)^2 + \left(\frac{\partial^2 X}{\partial y^2}\right)^2 dxdy \tag{9}$$

This expression is used in CAGD but without taking into account mechanical characteristics. [8] states that the minimisation of this energy can be characterised by $\Delta^2 X = 0$ for the thin plate energy. In this context, the *umbrella operator* is a discretisation of the Laplacian operator. Given a vertex P whose first neighbour ring is a set of vertices $(P_i)_{i \in 1..n}$:

$$U(P) = \frac{1}{n}\sum_{i=1}^{n} P_i - P \tag{10}$$

which leads to the following discretisation of the bilaplacian $U^2(P) = \frac{1}{n}\sum_{i=1}^{n} U(P_i) - U(P)$. Starting from this discretisation, the initial energy minimisation problem can thus be reduced to linear systems ($U^2(P)=0$ for any vertex P of the mesh). The authors showed that these systems could be solved by an iterative solving scheme. The computation involves the 2-neighbour ring of a summit.

In [32,33], a method of segmentation of a polyhedron using discrete invariants is developed. The formulas used result from [9]. The segmentation is first carried out by identifying the sharp edges of the polyhedron. They are detected by considerations on the values of the absolute and mean curvature. Each zone is then partitioned into sub zones of "homogeneous" curvature. In order to be completely disconnected from the shape of the triangles, the authors of [33] replace them by angular sectors with the same angles and a fixed arbitrary ray and adapt the formulae. These new values are called invariant of mean and Gaussian curvature. An invariant of absolute curvature is also defined. The obtained results show the validity of the approach.

In [25], an asymptotic analysis is proposed (the parameter of the discretisation h tends to 0) of the normal vector and Gaussian curvature estimates for three methods: paraboloid adjustment, angular defect (*classical formula*) and spherical image based on the ratio between a spherical image and the area of the triangles adjacent to P. The theoretical results state that these formulae are in $O(h)$ for scattered data (with only $O(1)$ for the classical formula and an additional assumption on the normal vector for the third method). For regular data, formulae become $O(h^2)$ approximations except for the classical formula which remains $O(1)$. However, the authors show that the *corrected formula* gives rise to an at least $O(h)$ approximation. Limits of this asymptotic study exist: it is of course useless when the data is a fixed grid that cannot be refined and in numerical applications, if h is not close enough from zero, it is not possible to use the equivalence results.

4. THE LOCAL APPROXIMATION APPROACH

In this case the idea is to consider that a vertex P and its neighbours are points on (or near) a continuous implicit surface, in general a quadric, but more complex case can be considered [14]. Once the surface has been fitted to this data, the local geometrical characteristics of the surface can be analytically computed and are considered as the discrete characteristics at vertex P. The computation of a mean curvature at P does not give rise to the difficulty exposed in the previous section. Moreover, another advantage is that the neighbours of a vertex do not require to be ordered: only the k nearest neighbours are specified. However, any artefact corresponding to discrete case cannot be handled, this is the main restriction that we discuss later.

Sander and Zucker's paper ([27]) is one of the first works dealing with this approach: the considered data is a space enumeration matrix (matrix of voxels) of noisy points. The normal vector at each point is first determined by applying a gradient technique on the *3D* image (first order approximation). A paraboloid is then locally fitted in the Darboux frame through a least squares technique taking into account the close points obtained during the initial estimate of the normal vector, and optionally their estimated normal vector. In this step lies actually the main difficulty of this approach that is, in finding a good estimate of this initial Darboux frame.

The paraboloid provides both principal curvatures and their associated principal directions except for an umbilical point. However, this direct approach does not lead to coherent values among different neighbours. Therefore, for each vertex, the principal directions and the normal vector are iteratively improved by comparing the results with those of its neighbours while adjusting them for a best local agreement.

This method is essentially a local one, but it is obviously influenced by the size of the considered neighbourhood. Depending on this size, the quadric either interpolates or approximates points. Assuming that the origin P of the frame is a point of the quadric, its characteristics are computed at the origin. Otherwise point P must be projected onto the quadric in order to find the point P^* where the characteristics are computed. These results are applied in [7] to plot lines of curvature for a surface given by a cloud of scattered points.

Douros and Buxton’s work ([18]) propose a second technique in order to get rid of the initial Darboux frame estimation. It consists in fitting the data with a general quadric determined by 10 coefficients $(ax^2 + by^2 + cz^2 + 2d \cdot xy + 2e \cdot xz + 2f \cdot yz + 2g \cdot x + 2h \cdot y + 2i \cdot z + j = 0)$.

The first problem met in this approach is that the corresponding system is homogeneous implying that an additional condition must be added to avoid finding all the coefficients equal to 0. Fixing any coefficient to a non-zero value implies that only a linear system must be solved ([31]). Even if the best coefficient to fix is probably j, such a choice introduces an arbitrary decision (both on the chosen coefficient and on its value). Another solution lies in fixing a condition on the 2-norm of the solution vector ([18]). The problem must then be solved under constraints.

Moreover, like in previous approach, if vertex P is not a point on the quadric (due to the choice of the approximation method), it must be projected onto it (on a point P^*) where the characteristics can be computed. This provides interesting results but requires additional works: further numerical analysis, choice of the best constraints and determination of an optimal number of neighbours. The main issue is that the method provides characteristics of a C^2 surface and is thus completely inefficient to detect any of the artefacts specific to the discrete case.

In [24], three methods are compared: a quadric fitting and two original methods. The principle of the first one, called circle method is very simple: starting from three points, whose middle point is the studied point, the circle going through these three points makes it possible to estimate the normal curvature in the direction of the tangent vector of the circle. With a certain number of well selected such triples, one can have a correct idea of the principal directions and principal curvatures applying Euler Theorem. The second method is more complex. Using locally cylindrical coordinates and assuming that one of the principal curvatures k is known, there exists an inversion locally transforming the surface into a cylinder. Mapping the cloud of points through this inversion corresponds to adjusting these points with a cylinder whose curvature h will provide the second principal curvature (k-h). This adjustment is a nonlinear process in which k can be considered as a variable rather than a datum. Both curvatures can thus be asymmetrically estimated. The second originality of this work is to compare these methods with a statistical approach for noisy data (initially on given surfaces): the standard deviations for the considered geometric invariants are plotted according to the variance of the introduced noise. The circle method is the fastest, but its limits are easily reached. The two other methods lead to close results. Although the proposed methods can be applied to any data, the data used in this paper are either on a regular grid simulating a laser scan with a constant z or on surfaces given by an equation $z=f(x,y)$, which probably skews the results. Another effect of these assumptions on the data is to remove the main difficulty when looking for the osculatory paraboloid: the estimate of the Darboux trihedron in which the surface is locally described. As a matter of fact, with these assumptions the study can be achieved directly in the global frame. The normal vector deduced from this paraboloid is used to initialize the third method. The authors note that this estimate is probably insufficient to obtain good results.

Similarly [29] numerically compares five different methods for local estimation of curvature geometric properties (both Gaussian and mean curvature), namely: paraboloid fitting, circular fitting, angular defect, the Wanatabe and Belyaev approach ([34]) and the Taubin ([30]) approach (these last two methods actually provide an estimate of the principal curvatures). All these algorithms are tested on meshes originating from NURBS surfaces and are compared with the analytically computed values of the Gaussian and mean curvatures of these surfaces. The best method for the estimation of the Gaussian curvature seems to be the angular defect approach whereas the paraboloid fitting provides the best estimates for the mean curvature. On the overall, the paraboloid fitting scheme seems to be the most stable method. However, these results only deal with NURBS surfaces, which are of course closer to paraboloid fitting.

5. A NEED FOR FINER TOOLS

The work of Alboul and Van Damme concerning the classification of the vertices according to discrete invariants is an interesting approach, allowing a first detection of singular vertices in meshes. Let us illustrate the interest of such a classification through the case of the well-known Schwarz lantern (actually a cylinder of radius R and height h). A possible triangulation of this surface is presented in figure 4 (left); let m be the number of parallels and n be the number of vertices on each parallel.

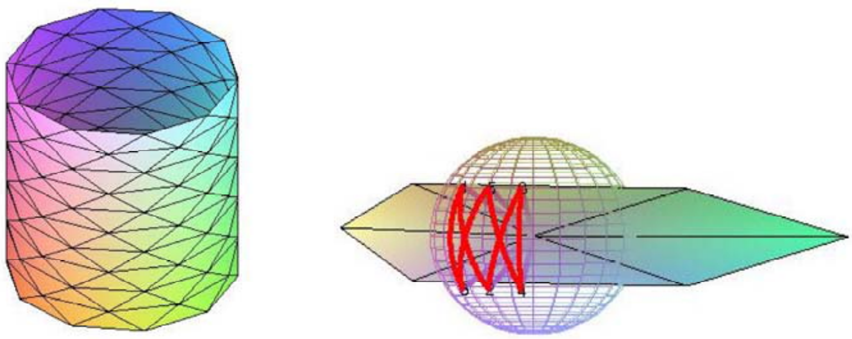

Fig. 4. A triangulation of the Schwarz lantern (left). Detail of a vertex and its Gauss map (right)

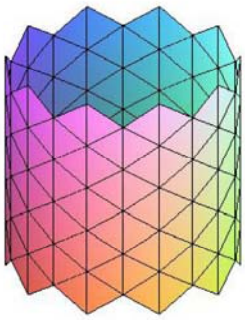

Fig. 5. The Schwarz lantern

All the vertices of such a triangulation are of type fan. Actually, one of the properties of this triangulation is that its area does not converge towards that of the cylinder unless n and m satisfy a very specific dependence relation. However, another triangulation of the same set of vertices is that of figure 5 whose vertices are convex and whose area converge towards that of the cylinder.

Unfortunately, other difficulties are encountered, contrary to what occurs for G^2 surfaces:

– At least six cases can be exhibited: convex, concave, saddle, convex-fan, concave-fan and saddle-fan. This second classification does not take into account the singular cases and the various limit cases to be considered for an exhaustive classification. We can in addition notice that it is possible to switch between neighbours from a convex vertex to a saddle vertex (figure 6, left).
– A convex vertex or its symmetrical concave has the same spherical indicatrix. It is however significant to distinguish them, in so far as one can encounter two neighbouring vertices in the discretisation, one (fan) convex, the other (fan) concave without an intermediate case (figure 6, right).

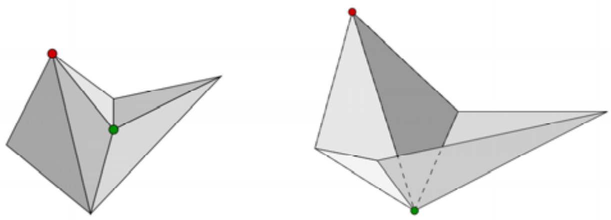

Fig. 6. Left: A convex vertex and its saddle neighbour. Right: A convex vertex and its concave-fan neighbour

- The mixed case initially suggested does not distinguish a convex-fan vertex from a saddle-fan vertex. However these two points are very different since the first has a supporting plane (and is obtained by addition of "reasonable" folds around a convex vertex) whereas the second does not have a supporting plane (and is obtained by addition of "reasonable" folds around a saddle vertex).
- A convex-fan vertex can have a negative angular defect as proposed in figure 7 (left). Moreover by increasing the number of vertices the angular defect is not bounded from below, as suggested in figure 7 (right). We can note that such vertices do not fulfil the conditions imposed in [11].

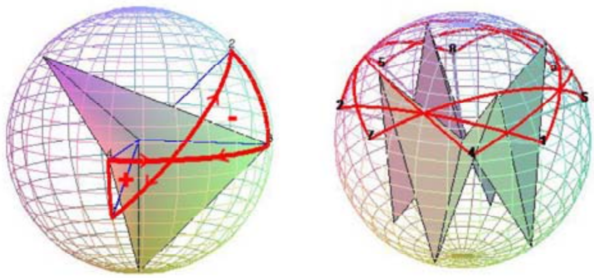

Fig. 7. A convex-fan vertex with a negative angular defect (left) and with a large negative angular defect (right)

These difficulties let us claim that the two approaches of discrete curvatures presented above do not completely lead to relevant estimators in any situation, even if they provide interesting results in many cases. The latter correspond to rather good discretisations with a good number of neighbours and relevant spatial distribution according to the radii of curvature. This is the reason why a local analysis of polyhedral surfaces is necessary. The first attempt is to provide warnings on the existence of “difficult” vertices and also define additional new indicators suitable for such vertices. In this frame the main idea is to consider systematically convexity properties of edges, characterised by mixed products with adjacent edges.

For that purpose, both polyhedrons are normalized so that they can be considered as spherical polygons: a standardised polyhedron associated with vertex P and its indicatrix. For a spherical polygon we note $t(v_i)$, the *type* of an edge v_i, which is the sign of determinant $|v_{i-1},v_i,v_{i+1}|$ (-1 for convex, +1 for concave). Two properties can directly be deduced from the definition: if n_i is the unit normal associated with a face spanned by a_i and a_{i+1}, we have

$$t(n_i) = t(a_i)t(a_{i+1}) \qquad (11)$$

$$\prod_i t(n_i) = \prod_i t(a_i)t(a_{i+1}) = \prod_i t(a_i)^2 > 0 \qquad (12)$$

The shape of the Gauss indicatrix, in the sense of the convexity-concavity of edges, is thus directly determined by the shape of polyhedron at vertex P. We deduce the types of normals by the rules *convex-convex or concave-concave gives concave normal* and *convex-concave or concave-convex gives convex normal*. This implies also that the number of convex edges of an indicatrix is even. This intuitively corresponds to the fact that symmetrical vertices, obtained by exchanging of convex and concave, give the same indicatrix. Roughly speaking, equivalent considerations let us claim that the number of types of vertices on the object is twice the number of types of vertices on the indicatrices. We can also easily deduce from these properties that a "pure" saddle vertex, with all convex normals, is only possible when its number of neighbours is even and that with four neighbours it is not possible to have a concave indicatrix (which has 3 convex edges and 1 concave edge).

To begin classification we define the shape of a vertex with k neighbours by a word of k letters, sequence of C (for a convex edge) and K (for a concave edge) and we list all the shapes following the numbers of convex and concave edges, from convex (all C) to concave (all K). In the case of a vertex with four neighbours we can list the shapes according to the number of convex edges of polyhedron/indicatrix: CCCC/KKKK convex, CCCK/KKCC convex-fan, CCKK/KCKC non-feasible (self-intersection), CKCK/CCCC saddle, CKKK/CKKC concave-fan, KKKK/KKKK concave.

The case CCKK, with indicatrix KCKC corresponds to a *pinch* vertex [1] and is excluded. This can easily be deduced by studying the relative positions of normal vectors and faces. The case of 5 neighbours can be obtained in the same way but, unfortunately, a more accurate study shows that there exist sub cases. This analysis requires additional investigations but is a first step towards a new vertex classification.

6. CONCLUSION

Among works dealing with discrete curvature, two approaches are mainly proposed: the use of angular defects and a local approximation. Even if the asymptotic behaviours are significant and must be taken into account, the reality of geometric modelling is a set of points, often without any possibility of refinement or improvement. Let us note, however, that the polyhedral approximation of a continuous surface makes it possible by subdivision to approach the asymptotic conditions. One must also take into account that structuring the set of vertices is not a simple problem. Delaunay triangulation is not necessarily the one providing the best representation of the shapes as noted in [19]. Finding the nearest neighbours of a vertex can also be a real challenge in extreme conditions. It appears significant under these conditions to be able to *a priori* analyse the scattered data to estimate the coherence of discrete calculations carried out. This is particularly accurate when an approximation by a quadric is used since this approach skips the potential artefacts associated with the discrete nature of the problem. Beyond these considerations and difficulties, many important applications in CAGD such as triangulation optimisation, segmentation, reverse engineering, image or surface analyses already provide interesting results.

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